Journal of Approximation Theory **111**, 180–195 (2001)
doi:10.1006/jath.2001.3571, available online at http://www.idealibrary.com on IDEAL®

Deformation of Minimal Polynomials and Approximation of Several Intervals by an Inverse Polynomial Mapping

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Communicated by Vilmos Totik

Received March 15, 1999; accepted in revised form February 20, 2001; published online June 20, 2001

In this paper we show that for a given set of l real disjoint intervals $E_l = \bigcup_{j=1}^{l} [a_{2j-1}, a_{2j}]$ and given $\varepsilon > 0$ there exists a real polynomial $\mathcal{T}$ and a set of l disjoint intervals $\tilde{E}_l = \bigcup_{j=1}^{l} [\tilde{a}_{2j-1}, \tilde{a}_{2j}]$ with $\tilde{E}_l \supseteq E_l$ and $\|(\tilde{a}_1, ..., \tilde{a}_{2l}) - (a_1, ..., a_{2l})\|_{\max} < \varepsilon$, such that $\mathcal{T}^{-1}([-1, 1]) = \tilde{E}_l$. The statement follows by showing how to get in a constructive way by a continuous deformation procedure from a minimal polynomial on E_l with respect to the maximum norm a polynomial mapping of $\tilde{E}_l$.

1. SOME BASIC FACTS ON T- AND MINIMAL POLYNOMIALS AND INVERSE POLYNOMIAL MAPPINGS

Let $l \in \mathbb{N}$, $a_1, a_2, ..., a_{2l} \in \mathbb{R}$, $a_1 < a_2 < \cdots < a_{2l}$, and let

$$E_l := \bigcup_{j=1}^{l} [a_{2j-1}, a_{2j}] \qquad \text{and} \qquad \mathcal{H}(x) := \prod_{j=1}^{2l} (x - a_j). \tag{1.1}$$

Note that $\mathcal{H}(x) \leqslant 0$ for $x \in E_l$. As usual, let $\mathbb{P}_n$, $n \in \mathbb{N}_0$, denote the set of real polynomials of degree less or equal n. If $p_n(x) = d_n x^n + \cdots \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ then $\hat{p}_n(x) = p_n(x)/d_n$ denotes the monic polynomial of degree n. A point $y \in E_l$ is called extremal point (abbreviated e-point) of $p \in \mathbb{P}_n$ on E_l if $|p(y)| = \|p\|_{E_l}$, where $\|p\|_{E_l} := \max_{x \in E_l} |p(x)|$. e-points from $\mathrm{Int}(E_l)$ and from ∂E_l are called interior and boundary e-points, respectively. The points $x_1, x_2, ..., x_m \in E_l$, $x_1 < x_2 < \cdots < x_m$, are called alternation points

[1] This work was supported by the Austrian Science Fund FWF, project-number P12985-TEC.

0021-9045/01 $35.00

(abbreviated a-points) of p on E_l if they are e-points of p and $p(x_j) = -p(x_{j+1})$ for $j=1, 2, ..., m-1$. Finally, a monic polynomial $\mathscr{M}_n(x) = x^n + \cdots$ is called minimal polynomial on E_l if

$$\min_{c_j \in \mathbb{R}} \|x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0\|_{E_l} = \|\mathscr{M}_n(x)\|_{E_l}.$$

Let us recall that by the Alternation-Theorem $\mathscr{M}_n$ is a minimal polynomial on E_l if and only if it has $n+1$ alternation points on E_l.

In this paper Chebyshev-polynomials (abbreviated T-polynomials) on E_l, which we define in the following way, will play an important role.

Definition 1.1. A polynomial $\mathscr{T}_n(x) = c_n x^n + \cdots \in \mathbb{P}_n$, $n \in \mathbb{N}$, $c_n \in \mathbb{R} \setminus \{0\}$, is called T-polynomial on E_l if it has $n+l$ e-points on E_l. A T-polynomial of degree n on E_l is denoted by $\mathscr{T}_n(x, E_l)$ where E_l is omitted if there is no danger of confusion. The polynomial $\tilde{\mathscr{T}}_n(x, E_l) = \mathscr{T}_n(x, E_l)/\|\mathscr{T}_n(x, E_l)\|_{E_l}$ is called normed T-polynomial on E_l.

Naturally, the classical T-polynomial of the first kind $T_n(x) = \cos n \arccos x$, $x \in [-1, 1]$, is a normed T-polynomial on $[-1, 1]$ for which the interior e-points are given by the zeros of $U_{n-1}(x) = \sin n \arccos x / \sin \arccos x$. T-polynomials on two intervals have been studied by Achieser [1, 3], see also [18], with the help of elliptic functions and by the author in [14, 15] with the help of orthogonal polynomials. The case of several intervals has been investigated by the author in [16, Section 2], [17], see also [25]. The reason why we called these polynomials T-polynomials is that they share many properties with the classical T-polynomials, as the next proposition shows.

Proposition 1.1 ([16, Section 2], [17]).

(i) *Let $\mathscr{T}_n$ be a T-polynomial on E_l, then $\mathscr{T}_n$ has the following properties*:

(i$_1$) *$\mathscr{T}_n$ has exactly $n-l$ interior e-points $z_1, z_2, ..., z_{n-l} \in \mathrm{Int}(E_l)$ and the $2l$ boundary e-points $a_1, a_2, ..., a_{2l} \in \partial E_l$. Moreover, $\mathscr{T}_n$ has $n+1$ a-points on E_l, thus the monic T-polynomial $\hat{\mathscr{T}}_n(x) = x^n + \cdots$ is also the minimal polynomial on E_l.*

(i$_2$) *$\mathscr{T}'_n$ has, besides the $n-l$ simple zeros $z_1, z_2, ..., z_{n-l} \in \mathrm{Int}(E_l)$, exactly one zero d_j in each gap (a_{2j}, a_{2j+1}), $j = 1, 2, ..., l-1$. Furthermore, $\mathscr{T}_n$ has n simple zeros in $\mathrm{Int}(E_l)$ and $E_l = \tilde{\mathscr{T}}_n^{-1}([-1, 1])$.*

(i$_3$) *If $\mathscr{T}_n$ is a T-polynomial on E_l, then $T_k(\tilde{\mathscr{T}}_n)$, $k \in \mathbb{N}$, are T-polynomials of degree kn on E_l. Furthermore, if there exists no other T-polynomial on E_l of degree less than n, then the polynomials $T_k(\tilde{\mathscr{T}}_n)$, $k \in \mathbb{N}$, are the only T-polynomials on E_l.*

(ii) *$\mathcal{T}_n$ is a T-polynomial on E_l if and only if there exists a polynomial $\mathcal{U}_{n-l} \in \mathbb{P}_{n-l}$ with $n-l$ simple zeros in* $\mathrm{Int}(E_l)$, *such that*

$$\mathcal{T}_n^2(x) = \mathcal{H}(x)\,\mathcal{U}_{n-l}^2(x) + L^2, \tag{1.2}$$

where $\mathcal{H}$ is defined in (1.1) *and $L \in \mathbb{R} \setminus \{0\}$.*

(iii) *If $\mathcal{T}_n(x) = x^n + \cdots \in \mathbb{P}_n$ is a T-polynomial on E_l, then*

$$\mathcal{T}_n'(x) = n\,\mathcal{U}_{n-l}(x)\,r_{l-1}(x), \qquad \text{where} \quad r_{l-1}(x) := \prod_{j=1}^{l-1} (x - d_j), \tag{1.3}$$

where $\mathcal{U}_{n-l}$ is given by (1.2) *and the d_j's are defined in* (i_2). *Furthermore, by* (1.2), *$\hat{\mathcal{U}}_{n-l}$ is the minimal polynomial of degree $n-l$ on E_l with respect to the weight function $\sqrt{-\mathcal{H}(x)}$.*

(iv) *Let $p_n(x) = c_n x^n + \cdots \in \mathbb{P}_n$, $c_n \neq 0$, be a polynomial with n simple real zeros and $K := \min\{|p_n(x)| : p_n'(x) = 0\}$, then, for every $\mu \in (0, K]$, $\mathcal{T}_{n,\mu}(x) := \frac{1}{\mu} p_n(x)$ is a T-polynomial on the set of intervals $\tilde{\mathcal{T}}_{n,\mu}^{-1}([-1, 1])$.*

(v) *There exists a T-polynomial $\mathcal{T}_n$ on E_l with $m_j + 1$ e-points on $[a_{2j-1}, a_{2j}]$, $j = 1, 2, ..., l$, if and only if there are l integers $m_j \in \{1, ..., n-l\}$ with $\sum_{j=1}^{l} m_j = n$, such that*

$$\frac{1}{\pi} \int_{a_{2j-1}}^{a_{2j}} \frac{|r_{l-1}(x)|}{\sqrt{|\mathcal{H}(x)|}}\, dx = \frac{m_j}{n}, \qquad j = 1, 2, ..., l, \tag{1.4}$$

where $r_{l-1}(x) = x^{l-1} + \cdots$ is the unique polynomial that satisfies

$$\int_{a_{2j}}^{a_{2j+1}} \frac{r_{l-1}(x)}{\sqrt{|\mathcal{H}(x)|}}\, dx = 0, \qquad j = 1, 2, ..., l-1. \tag{1.5}$$

Furthermore, if there exists a T-polynomial $\mathcal{T}_n$ on E_l, then $\tilde{\mathcal{T}}_n$ is given by

$$\tilde{\mathcal{T}}_n(z) = \cosh\left(n \int_{a_1}^{z} \frac{r_{l-1}(x)}{\sqrt{\mathcal{H}(x)}}\, dx\right), \tag{1.6}$$

where the polynomial r_{l-1} which satisfies (1.5) *is identical to the polynomial r_{l-1} from* (1.3).

In [16], T-polynomials have been defined by relation (1.2). Let us also mention that in the single interval case $E_1 = [-1, 1]$, relation (1.2) becomes the well known relation for classical T-polynomials of the first and second kind

$$T_n^2(x) + (x^2 - 1)\, U_{n-1}^2(x) = 1.$$

As already mentioned a monic T-polynomial $\mathcal{T}_n$ on E_l is a minimal polynomial on E_l which has the additional property that $|\mathcal{T}_n|>1$ on $\mathbb{R}\backslash E_l$. The precise connection between T- and minimal polynomials is given in part (ii) of the following proposition.

PROPOSITION 1.2. (i) *Let (n_k) be a subsequence of the natural numbers and let for $k\in\mathbb{N}$ $\mathcal{M}_{n_k}$ be the minimal polynomial on the compact set $K(n_k)\subseteq[a,b]$, $a<b$. Then within $K(n_k)$ the distance of two consecutive alternation points is at most $\mathcal{O}(1/n_k)$.*

(ii) *If $\mathcal{M}_n$ is a minimal polynomial on E_l then $\mathcal{M}_n$ is a T-polynomial on $l^*=l+l'$, $0\leqslant l'\leqslant l-1$, intervals $E^*_{l^*,n}$*

$$E^*_{l^*,n}=\bigcup_{j=1}^{l}\,[a_{2j-1,n},a_{2j,n}]\cup\bigcup_{\nu=1}^{l'}\,[c_{2j_\nu-1,n},c_{2j_\nu,n}],$$

$j_\nu\in\{1,...,l-1\}$ for $\nu=1,...,l'$, where $a_{1,n}=a_1$, $a_{2l,n}=a_{2l}$,

$$a_{2j}\leqslant a_{2j,n}<a_{2j+1,n}\leqslant a_{2j+1}\quad \textit{and}\quad a_{2j,n}=a_{2n}\ \textit{or}\ a_{2j+1,n}=a_{2j+1}$$

for $j=1,...,l-1$ and

$$a_{2j_\nu,n}=a_{2j_\nu}<c_{2j_\nu-1,n}<c_{2j_\nu,n}<a_{2j_\nu+1,n}=a_{2j_\nu+1}\quad \textit{for}\ \ \nu=1,...,l'.$$

Furthermore, on each "c-interval" $[c_{2j_\nu-1,n},c_{2j_\nu,n}]$, $\nu=1,...,l'$, $\mathcal{M}_n$ has no interior e-point and $\mathcal{M}_n$ is a minimal polynomial on every subset of $E^*_{l^*,n}$ which contains E_l. Finally, if we write $E^*_{l^*,n}=E_{l,n}\cup C_{l',n}$ then

$$E_{l,n}\xrightarrow[n\to\infty]{}E_l\qquad \textit{and}\qquad \lim_{n\to\infty}\lambda(C_{l',n})=0,$$

where λ denotes the Lebesgue-measure.

Proof. Part (i) follows by slightly modifying the proof of Theorem 1 in [10]. Indeed, in the first part of the proof replace $[a,b]$ by $K(n_k)$ and let α and β be two consecutive alternation points of $\mathcal{M}_{n_k}$ within $K(n_k)$. Note that in the case under consideration $r=k_n=s_n=0$. Then it follows that the minimum deviation of f, f defined in [10], with respect to $\mathbb{P}_{n_k}$ on $K(n_k)$ is $\geqslant 1$. Hence, f extended to $[a,b]$ by $f\equiv 0$ on $[a,b]\backslash K(n_k)$ has also minimum deviation $\geqslant 1$ on $[a,b]$ with respect to $\mathbb{P}_{n_k}$. Now proceeding word by word as in the last lines of the proof of Theorem 1 the assertion follows.

Up to the limit relations part (ii) follows immediately with the help of the Alternation-Theorem and by a careful counting of the zeros of the derivative of $\mathcal{M}_n$ as already mentioned in [16]. Both limit relations follow immediately from part (i). ∎

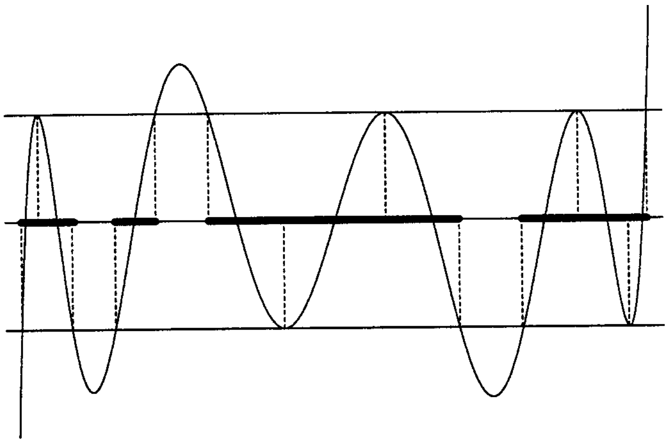

FIG. 1.1. T-polynomial of degree $n=9$ on $l=4$ intervals, which is also a minimal polynomial on the union of the first, third and fourth interval.

The reason why minimal polynomials are so difficult to handle is that "c-intervals" may appear, see Fig. 1.1, where the number and location will in general vary with respect to n. Let us give a simple example.

EXAMPLE 1.1. Let $E_2(b)=[-1,-b]\cup[b,1]$ for $b\in(0,1)$. Then by symmetry arguments and the Alternation Theorem it follows that for $b\in(0,1)$ $\mathcal{M}_{2n}(x,E_2(b))=\widehat{T_n(\tilde{\mathcal{T}}_2(x))}$, where $\tilde{\mathcal{T}}_2(x)=\frac{2x^2-b^2-1}{1-b^2}$, is a minimal and a T-polynomial on $E_2(b)$, i.e., for even n no c-interval appears. Again by symmetry arguments it can be shown that $\mathcal{M}_{2n+1}(x,E_2(b))$ is an odd polynomial which is a T-polynomial on $[-1,-b]\cup[c_{1,n},c_{2,n}]\cup[b,1]$, where $c_{1,n}=-c_{2,n}$ and $0<c_{2,n}<b<1$. By the way, it is possible to relate $\mathcal{M}_{2n+1}$ to the Zolotareff-polynomial (see [15]) and to explicitly express $c_{2,n}$ in terms of elliptic functions ([11]).

Next let us discuss the connection of T-polynomials with inverse polynomial mappings.

Notation. As usual, we say that E_l is the inverse image of $[-1,1]$ under a polynomial mapping if there exists a polynomial P with complex coefficients such that $E_l=P^{-1}([-1,1])$.

The connection with T-polynomials is now the following.

PROPOSITION 1.3. *E_l is the inverse image of $[-1,1]$ under the polynomial mapping $\tilde{\mathcal{T}}_n$ if and only if $\tilde{\mathcal{T}}_n$ is a normed T-polynomial on E_l.*

Proof. The sufficiency part of the proof follows immediately from point (i_2) of Proposition 1.1. For the necessity part see [20, Proposition 2]. ∎

Inverse polynomial images of arbitrary compact sets K of the complex plane, in particular of the interval $[-1, 1]$, have been studied in [6–8, 19, 20, 22]. One of the main reasons why these sets are of foremost interest is that frequently properties on K are inherited in a suitable way to the inverse image $\mathscr{T}_n^{-1}(K)$. So certain extremal problems are easy to handle on an inverse polynomial image of $[-1, 1]$ (for instance if p_j is an $L_q(\mu)$-minimal polynomial on $[-1, 1]$, $q \in [1, \infty]$, then $p_j \circ \mathscr{T}_n$ is an $L_q(\mu_{\mathscr{T}_n})$-minimal polynomial on $\mathscr{T}_n^{-1}([-1, 1])$, see [22] for details), but not on an arbitrary set of intervals. Therefore the question arises whether a given set of intervals can be approximated arbitrarily well by polynomial inverse images. In order to be able to treat this question we need some facts from potential theory (see e.g. [9, 27]).

Let $G(z, \infty) = g(z, \infty) + i\tilde{g}(z, \infty)$ be the complex Green function of $G = \mathbb{C} \cup \{\infty\} \setminus E_l$ with pole at ∞, i.e., $g(\cdot, \infty)$ is harmonic in $G \setminus \{\infty\}$ with a behaviour at ∞ given by

$$g(z, \infty) = \ln|z| + \text{harmonic function}$$

and

$$\lim_{G \ni z \to x} g(z, \infty) = 0 \qquad \text{for} \quad x \in E_l,$$

where $\tilde{g}(z, \infty)$ is a harmonic conjugate of g. It is well known that $g(z, \infty)$ can be represented in the form

$$g(z, \infty) = \int_{E_l} \ln|z - x| \, d\nu_{E_l}(x) - \ln \operatorname{cap}(E_l),$$

where ν_{E_l} is the so-called equilibrium (Frostman) measure of E_l and $\operatorname{cap}(E_l)$ is the capacity of E_l. Recall (see e.g. [24, Section III]) that

$$\lim_{n \to \infty} \|\mathscr{M}_n(x, E_l)\|_{E_l}^{1/n} = \operatorname{cap}(E_l), \tag{1.7}$$

where $\mathscr{M}_n(x, E_l)$ is the monic minimal polynomial on E_l, and that (1.7) implies that the zero counting measure of $\mathscr{M}_n(x, E_l) = \prod_{j=1}^{n} (x - x_{j,n})$ converges in the weak star sense to the equilibrium measure of E_l, i.e.,

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j,n}} \xrightarrow[n \to \infty]{*} \nu_{E_l}, \tag{1.8}$$

where $\delta_{x_{j,n}}$ denotes, as usual, the Dirac-Delta measure at the point $x_{j,n}$. Furthermore it is known (see [28, Section 14]) that the complex Green function of E_l is of the form

$$G(z, \infty) = \int_{a_1}^{z} \frac{r_{l-1}(\xi)}{\sqrt{\prod_{j=1}^{2l} (\xi - a_j)}} \, d\xi, \tag{1.9}$$

where the integration is performed along a path in the complex plane cut along E_l and where $r_{l-1}(\xi) = \xi^{l-1} + \cdots \in \mathbb{P}_{l-1}$ is that unique polynomial which satisfies

$$\int_{a_{2j}}^{a_{2j+1}} \frac{r_{l-1}(x)}{\sqrt{\prod_{j=1}^{2l} (x - a_j)}} \, dx = 0 \qquad \text{for} \quad j = 1, ..., l-1. \tag{1.10}$$

Hence

$$\int_{E_l} \frac{d\nu(x)}{z - x} = G'(z, \infty) = \frac{r_{l-1}(z)}{\sqrt{\mathscr{H}(z)}},$$

which gives with the help of the Sochozki–Plemelj formula that the equilibrium measure ν for E_l is given by (see [13, Lemma 1])

$$\nu'(x) = \begin{cases} |r_{l-1}(x)|/\pi \sqrt{-\mathscr{H}(x)} & \text{for} \quad x \in E_l \\ 0 & \text{elsewhere.} \end{cases} \tag{1.11}$$

Let us note that condition (1.10) implies that r_{l-1} has exactly one zero in each gap (a_{2j}, a_{2j+1}), $j = 1, ..., l-1$. We also would like to point out that the polynomial r_{l-1} satisfying condition (1.10) is known if there is a T-polynomial $\mathscr{T}_n$ on E_l. Indeed, by Proposition 1.1(v) it is the polynomial r_{l-1} from (1.3).

Now let us recall that the following relation holds between the equilibrium measure and the harmonic measure for $\overline{\mathbb{C}} \backslash E_l$ (see e.g. [23, Thm. 4.3.14]):

$$\nu(B) = \omega(\infty, B, \overline{\mathbb{C}} \backslash E_l) \qquad \text{for any Borel subset } B \text{ of } E_l. \tag{1.12}$$

As usual, $\omega(z, B, \overline{\mathbb{C}} \backslash E_l)$, $z \in \overline{\mathbb{C}} \backslash E_l$, denotes the harmonic measure for $\overline{\mathbb{C}} \backslash E_l$ of $B \subseteq E_l$ which is that harmonic and bounded function on $\overline{\mathbb{C}} \backslash E_l$ which satisfies for $\zeta \in E_l$ that $\lim_{z \to \zeta} \omega(z, B, \overline{\mathbb{C}} \backslash E_l) = i_B(\zeta)$, where i_B denotes, as usual, the characteristic function of B.

Thus (v) in Proposition 1.1 can be expressed in the following way. There exists a T-polynomial $\mathscr{T}_n$ on E_l with m_j+1 e-points on $[a_{2j-1}, a_{2j}]$, $j=1, ..., l$, if and only if there are l natural numbers $m_j \in \{1, 2, ..., n-l\}$ with $\sum_{j=1}^{l} m_j = n$ such that the harmonic measure satisfies

$$\omega(\infty, [a_{2j-1}, a_{2j}], \overline{\mathbb{C}} \backslash E_l) = \frac{m_j}{n} \qquad \text{for} \quad j=1, ..., l.$$

This form of the Proposition has been proved in [4] with the help of deep results of Widom [28] on the asymptotics of orthogonal polynomials.

LEMMA 1.1. *Let $E_l \subseteq \tilde{E}_l$ and let us assume that $[a_{2j^*-1}, a_{2j^*}]$ is a component of both E_l and $\tilde{E}_l$ and that $\tilde{E}_l \backslash E_l \supseteq [c, d]$, $c<d$. Then $\nu_{E_l}([a_{2j^*-1}, a_{2j^*}]) - \nu_{\tilde{E}_l}([a_{2j^*-1}, a_{2j^*}]) > 0$.*

Proof. Let us consider the two harmonic measures $\omega(z, [a_{2j^*-1}, a_{2j^*}], \overline{\mathbb{C}} \backslash E_l)$ and $\omega(z, [a_{2j^*-1}, a_{2j^*}], \overline{\mathbb{C}} \backslash \tilde{E}_l)$. Since $\tilde{E}_l \backslash E_l \supseteq [c, d]$, we have by the Carleman extension principle (see e.g. [12, IV, Section 2])

$$\omega(z, [a_{2j^*-1}, a_{2j^*}], \overline{\mathbb{C}} \backslash E_l) - \omega(z, [a_{2j^*-1}, a_{2j^*}], \overline{\mathbb{C}} \backslash \tilde{E}_l) > 0 \tag{1.13}$$

on each compact subset of $\overline{\mathbb{C}} \backslash \tilde{E}_l$. Considering (1.13) at the point $z=\infty$ and recalling relation (1.12) the lemma is proved. ∎

In the following let $\#\mathscr{E}(\mathscr{M}_n, B)$, $B \subseteq E_l$, denote the number of e-points of $\mathscr{M}_n(\cdot, E_l)$ on B. With the help of Lemma 1.1 we obtain

PROPOSITION 1.4. *Let (n_k) be a strictly monotone subsequence of the natural numbers. Suppose that for each $k \in \mathbb{N}$ $E_l \subseteq E_{l, n_k} = \bigcup_{j=1}^{l} [a_{2j-1, n_k}, a_{2j, n_k}] \subseteq \tilde{E}_{l, n_k} = \bigcup_{j=1}^{l} [\tilde{a}_{2j-1, n_k}, \tilde{a}_{2j, n_k}]$ with $a_1 = a_{1, n_k} = \tilde{a}_{1, n_k} < a_2 \leqslant a_{2, n_k} \leqslant \tilde{a}_{2, n_k} < \tilde{a}_{3, n_k} \leqslant a_{3, n_k} \leqslant a_3 < a_4 \leqslant a_{4, n_k} \leqslant \tilde{a}_{4, n_k} < \cdots < \tilde{a}_{2l-1, n_k} \leqslant a_{2l-1, n_k} \leqslant a_{2l-1} < a_{2l} = a_{2l, n_k} = \tilde{a}_{2l, n_k}$, that $\tilde{E}_{l, n_k}$ and E_{l, n_k} have a common component $[a_{2j^*-1, n_k}, a_{2j^*, n_k}] = [\tilde{a}_{2j^*-1, n_k}, \tilde{a}_{2j^*, n_k}]$ and that $E_{l, n_k} \xrightarrow[k \to \infty]{} E_l$. Furthermore let us assume that for all $k \in \mathbb{N}_0$ the minimal polynomials $\mathscr{M}_{n_k}$ and $\tilde{\mathscr{M}}_{n_k}$ on E_{l, n_k} and $\tilde{E}_{l, n_k}$, respectively, have the property that*

$$\begin{aligned} &\#\mathscr{E}(\tilde{\mathscr{M}}_{n_k}, [\tilde{a}_{2j-1, n_k}, \tilde{a}_{2j, n_k}]) - \#\mathscr{E}(\mathscr{M}_{n_k}, [a_{2j-1, n_k}, a_{2j, n_k}]) \\ &\qquad \leqslant const \qquad \textit{for} \quad j=1, ..., l. \end{aligned}$$

Then the following statement holds:

$$\tilde{E}_{l, n_k} \xrightarrow[k \to \infty]{} E_l. \tag{1.14}$$

Proof. Let us suppose that there is a subsequence of (n_k) denoted by (n_k) again such that $\tilde{E}_{l,n_k} \underset{k\to\infty}{\not\longrightarrow} E_l$, that is, if necessary by taking another subsequence of (n_k), that

$$\tilde{E}_{l,n_k} \to \tilde{E}_l = \bigcup_{j=1}^{l} [\tilde{a}_{2j-1}, \tilde{a}_{2j}] \qquad \text{where} \quad \tilde{E}_l \backslash E_l \supseteq [c, d], \quad c<d.$$

Then $\tilde{E}_l$ and E_l have the common component $[a_{2j^*-1}, a_{2j^*}]$ and thus by Lemma 1.1

$$0 < \nu_{E_l}([a_{2j^*-1}, a_{2j^*}]) - \nu_{\tilde{E}_l}([a_{2j^*-1}, a_{2j^*}]). \tag{1.15}$$

On the other hand it is known (see [24, Section III]) that

$$\lim_{k\to\infty} \|\mathscr{M}_k\|_{E_{l,n_k}}^{1/n_k} = \operatorname{cap}(E_l) \quad \text{and} \quad \lim_{k\to\infty} \|\tilde{\mathscr{M}}_{n_k}\|_{\tilde{E}_{l,n_k}}^{1/n_k} = \operatorname{cap}(\tilde{E}_l) \tag{1.16}$$

and that (1.16) implies that the zero counting measure of $\mathscr{M}_{n_k}$ and $\tilde{\mathscr{M}}_{n_k}$ converges in the weak star sense to the equilibrium measure of E_l and $\tilde{E}_l$, respectively. More precisely, if $\mathscr{M}_{n_k}(x) = \prod_{j=1}^{n_k} (x - x_{j,n_k})$ and $\tilde{\mathscr{M}}_{n_k}(x) = \prod_{j=1}^{n_k} (x - \tilde{x}_{j,n_k})$, then

$$\frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{x_{j,n_k}} \xrightarrow[k\to\infty]{*} \nu_{E_l} \qquad \text{and} \qquad \frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{\tilde{x}_{j,n_k}} \xrightarrow[k\to\infty]{*} \nu_{\tilde{E}_l}, \tag{1.17}$$

where δ_{x_j} is the Dirac-Delta measure at the point x_j. Since the zero distribution is the same as that of the e-points it follows from (1.17) and the assumption on the e-points that

$$\nu_{E_l}([a_{2j-1}, a_{2j}]) = \nu_{\tilde{E}_l}([\tilde{a}_{2j-1}, \tilde{a}_{2j}]) \qquad \text{for} \quad j = 1, ..., l.$$

But this is a contradiction to (1.15). ∎

By the suggestions of the referee the proof of Proposition 1.4 became essentially nicer and shorter.

Let us point out that it is essential that $\tilde{E}_{l,n_k}$ and E_{l,n_k} have a common component as the Example 1.1 shows. Put $E_{l,2n} = E_l = E_2(b)$ and $\tilde{E}_{l,2n} = E_2(b')$ with $0 < b < b'$ there and observe that the even minimal polynomials $\mathscr{M}_{2n}$ and $\tilde{\mathscr{M}}_{2n}$ have exactly $n+1$ e-points in $[b, 1]$ and $[b', 1]$, respectively.

2. MAIN RESULT

Now we are ready to treat the question whether the set of l intervals on which there exists a T-polynomial is dense in the set of l intervals.

For $l=2$ the denseness has been proved by Achieser [2] with the help of elliptic functions and later by Atlestam [5] by showing that for given a_3

$$h(a_2)=\int_{-1}^{a_2} \frac{|r_1(x)|}{\sqrt{(1-x^2)(x-a_2)(x-a_3)}}\, dx$$

is a bijective mapping on $(-1, a_3)$. Both methods turned out to be unsuitable for the proof of the general case of l intervals.

We shall give a constructive proof by showing how to get by a continuous deformation procedure from a minimal polynomial on E_l a T-polynomial on $\tilde{E}_l$, i.e., the desired polynomial mapping. The procedure is based on the following proposition.

PROPOSITION 2.1. *Let $\mathcal{T}_n$ be a T-polynomial on $\bigcup_{j=1}^{l} [a_{2j-1}, a_{2j}]$ and let $j^* \in \{2, 3, ..., l-1\}$. Then there exists an $\alpha^*>0$ such that there are strictly monotone increasing functions $a_{2j}(\alpha)$: $[0, \alpha^*] \to [a_{2j}, a_{2j+1}]$ for $j=1, 2, ..., j^*-1$, and $a_{2j^*-1}(\alpha)$: $[0, \alpha^*] \to [a_{2j^*-1}, a_{2j^*+1})$, as well as strictly monotone decreasing functions $a_{2j-1}(\alpha)$: $[0, \alpha^*] \to [a_{2j-2}, a_{2j-1}]$ for $j=j^*+1, ..., l$ such that the following statements hold*:

(i) *For each $\alpha \in [0, \alpha^*]$ there exists a T-polynomial $\mathcal{T}_n(x, \alpha)$ on*

$$E_l(\alpha)=\bigcup_{j=1}^{j^*-2} [a_{2j-1}, a_{2j}(\alpha)] \cup [a_{2j^*-3}, a_{2j^*-2}]$$

$$\cup [a_{2j^*-1}(\alpha), a_{2j^*}+\alpha] \cup \bigcup_{j=j^*+1}^{l} [a_{2j-1}(\alpha), a_{2j}].$$

Furthermore, for $\alpha \in [0, \alpha^*)$ $E_l(\alpha)$ consists of l disjoint intervals and $\mathcal{T}_n(x, \alpha)$ has on the first, second, third, ... interval of $E_l(\alpha)$ the same number of e-points as $\mathcal{T}_n(x)$ on the first, second, third, ... interval of E_l and all interior e-points which are smaller resp. larger than a_{2j^*} increase resp. decrease with respect to α.

(ii) *For $\alpha=\alpha^*$ there exists a $\mu \in \{1, 2, ..., j^*-2\}$ or a $\mu \in \{j^*+1, ..., l\}$ such that $a_{2\mu}(\alpha^*)=a_{2\mu+1}$, $\mu \in \{1, 2, ..., j^*-2\}$, or $a_{2\mu-1}(\alpha^*)=a_{2\mu-2}$, $\mu \in \{j^*+2, ..., l\}$, or $a_{2j^*+1}(\alpha^*)=a_{2j^*}+\alpha^*$, i.e., for $\alpha=\alpha^*$ at least one gap is closed. Furthermore, no interior e-points coalesce.*

Proof. The proof runs in a similar way as the proof of Theorem 2.9 in [21]. Let us prove the assertion for small $\alpha>0$ first. Let $\mathbf{d}^0=(d_1^0, ..., d_{n-1}^0)$, where $\{d_j^0\}_{j=1}^{n-1}$ consists of all interior e-points of $\mathcal{T}_n$ and the $l-1$ boundary points $a_2, a_4, ..., a_{2(j^*-2)}, a_{2j^*-1}, a_{2j^*+1}, ..., a_{2l-1}$. Furthermore let

$$\mathbf{c}=(a_1, a_3, ..., a_{2j^*-5}, a_{2j^*-3}, a_{2j^*-2}, a_{2j^*}+\alpha, a_{2j^*+2}, a_{2j^*+4}, ..., a_{2l}) \in \mathbb{R}^{l+1}$$

and for $\alpha = 0$ let us denote this vector by $\mathbf{c}^0$. Then it follows by Theorem 2.7 of [21] that for sufficiently small $\alpha > 0$ there is a vector

$$\mathbf{d}(\mathbf{c}) = (d_1(\mathbf{c}), ..., d_{n-1}(\mathbf{c}))$$

with $\mathbf{d}(\mathbf{c}^0) = \mathbf{d}^0$ such that $E(\mathbf{b}) = \bigcup_{j=1}^{n} [b_{2j-1}, b_{2j}]$, where $\{b_j\}_{j=1}^{2n} = \{c_\nu\}_{\nu=1}^{l+1} \cup \{d_\kappa(\mathbf{c})\}_{\kappa=1}^{n-1}$, $b_1 < b_2 \leqslant \cdots \leqslant b_{2n-1} < b_{2n}$, consists of l disjoint intervals and on $E(\mathbf{b}) =: E_l(\alpha)$ there exists a T-polynomial $\mathscr{T}_n(x, \mathbf{b}) =: \mathscr{T}_n(x, \alpha)$ which has the same number of e-points on each of the l disjoint intervals of $E_l(\alpha)$ as $\mathscr{T}_n$ on the corresponding intervals of E_l.

Concerning the monotonicity let us first note that besides the $n-l$ interior e-points of $\mathscr{T}_n(x, \alpha)$ all $l-1$ nonfixed boundary points $a_2(\alpha)$, $a_4(\alpha)$, ..., $a_{2(j^*-2)}(\alpha)$, $a_{2j^*-1}(\alpha)$, $a_{2j^*+1}(\alpha)$, ..., $a_{2l-1}(\alpha)$ are contained in $\{d_j(\mathbf{c})\}_{j=1}^{n-1}$. Now we claim that for each $d_\kappa(\mathbf{c})$ from $\{d_j(\mathbf{c})\}_{j=1}^{n-1}$ we have

$$\operatorname{sgn} \mathscr{T}_n(d_\kappa(\mathbf{c}), \alpha) = -\operatorname{sgn} \prod_{\substack{j=1 \\ j \neq \kappa}}^{n-1} (d_\kappa(\mathbf{c}) - d_j(\mathbf{c})). \tag{2.1}$$

Indeed, if $d_\kappa(\mathbf{c}) \geqslant a_{2j^*-1}(\alpha)$ then (2.1) follows from the observation that

$$\operatorname{sgn} \mathscr{T}_n(d_\kappa(\mathbf{c}), \alpha) = (-1)^{1+n_\kappa^+},$$

where n_κ^+ denotes the number of $d_j(\mathbf{c})'s$ which lie at the right hand side of $d_\kappa(\mathbf{c})$. If $d_\kappa(\mathbf{c}) < a_{2j^*-2}$ then

$$\operatorname{sgn} \mathscr{T}_n(d_\kappa(\mathbf{c}), \alpha) = (-1)^{n-(1+n_\kappa^-)},$$

where n_κ^- denotes the number of $d_j(\mathbf{c})'s$ which lie at the left side of $d_\kappa(\mathbf{c})$. This proves (2.1).

Next let

$$c_{j^*+1} = a_{2j^*} + \alpha$$

be the (j^*+1)th component of $\mathbf{c}$. Then we have by the relations (2.19) and (2.20) of [21] and (2.1) that

$$\begin{aligned} \frac{\partial d_\kappa}{\partial c_{j^*+1}}(\mathbf{c}) &= -\operatorname{sgn}\left(\frac{\mathscr{T}_n(a_{2j^*}+\alpha, \alpha)}{\mathscr{T}_n(d_\kappa(\alpha), \alpha)} \cdot \prod_{\substack{j=1 \\ j\neq\kappa}}^{n-1} \frac{a_{2j^*}+\alpha - d_j(\mathbf{c})}{d_\kappa(\mathbf{c}) - d_j(\mathbf{c})}\right) \\ &= \operatorname{sgn}(a_{2j} + \alpha - d_\kappa(\mathbf{c})), \end{aligned} \tag{2.2}$$

where in the last equality we have used the fact that

$$\operatorname{sgn} \mathscr{T}_n(a_{2j^*}+\alpha, \alpha) = \prod_{j=1}^{n-1} (a_{2j^*}+\alpha - d_j(\mathbf{c})),$$

which proves the monotonicity property. Thus the theorem is proved for small $\alpha > 0$.

The remaining part of the proof runs now in almost the same way as the proof of Theorem 2.9 in [21], more precisely as the part beginning three lines before relation (2.27). Indeed, since by [21, Corollary 2.5] the existence of the T-polynomial $\mathscr{T}_n(x, \alpha)$ is equivalent to

$$\int_{E_l(\alpha)} x^k \operatorname{sgn}\{p_{n-1}^{(\alpha)}(x)\}\, dx = 0, \qquad k = 0, ..., n-2, \tag{2.3}$$

where

$$p_{n-1}^{(\alpha)}(x) = \prod_{j=1}^{n-1} (x - d_j(\mathbf{c})),$$

it follows as in [21] that no interior e-point can coincide with another e-point as $\alpha \to \delta$ because then we could chose a $q \in \mathbb{P}_{n-2}$ such that

$$\int_{E_l(\delta)} q(x) \operatorname{sgn}\{p_{n-1}^{(\delta)}(x)\}\, dx > 0$$

which contradicts (2.3) for $\alpha = \delta$. Thus Theorem 2.7 of [21] can be applied successively as long as no boundary points coalesce, which gives the assertion. ∎

Now let us state and prove the announced denseness theorem.

THEOREM 2.1. *Let $E_l = \bigcup_{j=1}^{l} [a_{2j-1}, a_{2j}]$, $a_1 < a_2 < \cdots < a_{2l}$, be given. Then for every $\varepsilon > 0$ there exists a real polynomial $\mathscr{T}_n$ such that $\mathscr{T}_n^{-1}([-1, +1]) = \bigcup_{j=1}^{l} [\tilde{a}_{2j-1}, \tilde{a}_{2j}]$ and $\|(\tilde{a}_1, ..., \tilde{a}_{2l}) - (a_1, ..., a_{2l})\|_{\max} < \varepsilon$.*

Proof. By Proposition 1.2(ii) we may assume w.l.o.g. (if necessary we take a subsequence) that for all $n \geqslant n_0$ $\mathscr{M}_n$ is a minimal polynomial on E_l and a T-polynomial on

$$\begin{aligned} E_{l,n} \cup C_{l',n} &= \bigcup_{j=1}^{l} [a_{2j-1,n}, a_{2j,n}] \cup \bigcup_{j=1}^{l'} [c_{2j_\nu-1,n}, c_{2j_\nu,n}] \\ &= [a_1, a_{2,n}] \cup \cdots \cup [a_{2j_1-1,n}, a_{2j_1,n}] \cup [c_{2j_1-1,n}, c_{2j_1,n}] \\ &\quad \cup [a_{2j_1+1,n}, a_{2j_1+2,n}] \cup \cdots \cup [a_{2j_\nu-1,n}, a_{2j_\nu,n}] \\ &\quad \cup [c_{2j_\nu-1,n}, c_{2j_\nu,n}] \cup [a_{2j_\nu+1,n}, a_{2j_\nu+2,n}] \cup \cdots, \end{aligned}$$

where $E_{l,n} \supseteq E_l$ with $E_{l,n} \xrightarrow[n \to \infty]{} E_l$ and $\mathcal{M}_n$ is a minimal polynomial on $E_{l,n}$. In view of Proposition 2.1 there exists for each $n \geqslant n_0$ an $\alpha_{1,n} > 0$ such that for each $\alpha \in [0, \alpha_{1,n})$ there is a T-polynomial $\mathcal{T}_n(x, \alpha)$ on

$$\begin{aligned} E(\alpha) = [a_1, a_{2,n}(\alpha)] \cup \cdots \cup [a_{2j_1-3,n}, a_{2j_1-2,n}(\alpha)] \cup [a_{2j_1-1,n}, a_{2j_1,n}] \\ \cup [c_{2j_1-1,n}(\alpha), c_{2j_1,n} + \alpha] \cup [a_{2j_1+1,n}(\alpha), a_{2j_1+2,n}] \cup \cdots \\ \cup [a_{2j_\nu-1,n}(\alpha), a_{2j_\nu,n}] \cup [c_{2j_\nu-1,n}(\alpha), c_{2j_\nu,n}] \\ \cup [a_{2j_\nu+1,n}(\alpha), a_{2j_\nu+2,n}] \cup \cdots \cup [a_{2l-1,n}(\alpha), a_{2l}], \end{aligned}$$

which has the same number of e-points on each a- and c-interval. Note that we have fixed both endpoints of the a-interval preceding the first c-interval $[c_{2j_1-1,n}, c_{2j_1,n}]$, the left boundary point of each interval lying to the left hand side of $[a_{2j_1-1,n}, a_{2j_1,n}]$ and the right boundary point of each interval lying to the right hand side of $[c_{2j_1-1,n}, c_{2j_1,n}]$ (and study what happens for increasing α). By Proposition 2.1 the not fixed boundary points $a_{2j,n}(\alpha)$, $j = 1, ..., j_1 - 1$, at the left hand side of $a_{2j_1-1,n}$ are increasing and the not fixed boundary points at the right hand side of $c_{2j_1,n}$ are decreasing with respect to α. Hence, the following cases may occur for $\alpha = \alpha_{1,n}$.

Case 1: $c_{2j_1,n} + \alpha_{1,n} = a_{2j_1+1,n}(\alpha_{1,n})$. In this case the gap between $[c_{2j_1-1,n}(\alpha), c_{2j_1,n} + \alpha]$ and $[a_{2j_1+1,n}(\alpha), a_{2j_1+2,n}]$ becomes closed for $\alpha = \alpha_{1,n}$. For simplicity of notation let us first assume that no other gaps close, i.e., $c_{2j_\nu,n} < a_{2j_\nu+1,n}(\alpha_{1,n})$ and $c_{2j_\nu-1,n}(\alpha_{1,n}) > a_{2j_\nu,n}$ for $\nu = 2, ..., l'$. Hence, $E(\alpha_{1,n})$ is the form

$$\begin{aligned} E(\alpha_{1,n}) &= \bigcup_{j=1}^{j_1-1} [a_{2j-1,n}, a_{2j,n}(\alpha_{1,n})] \cup [a_{2j_1-1,n}, a_{2j_1,n}] \\ &\qquad \cup [c_{2j_1-1,n}(\alpha_{1,n}), a_{2j_1+2,n}] \cup \bigcup_{j=j_1+2}^{l} [a_{2j-1,n}(\alpha_{1,n}), a_{2j,n}] \\ &\qquad \cup \bigcup_{\nu=2}^{l'} [c_{2j_\nu-1,n}(\alpha_{1,n}), c_{2j_\nu,n}] \\ &=: E^1_{l,n} \cup C^1_{l'-1,n}, \end{aligned} \tag{2.4}$$

where $C^1_{l'-1,n} = \bigcup_{\nu=2}^{l'} [c_{2j_\nu-1,n}(\alpha_{1,n}), c_{2j_\nu,n}]$. Note that $E^1_{l,n} \supseteq E_{l,n}$ and that $E^1_{l,n}$ and $E_{l,n}$ have the common component $[a_{2j_1-1,n}, a_{2j_1,n}]$. Furthermore, by Proposition 2.1 again, $\mathcal{T}_n(x, \alpha_{1,n})$ is a T-polynomial on $E(\alpha_{1,n})$ and a minimal polynomial on $E^1_{l,n}$ which has on the interval $[c_{2j_1-1,n}(\alpha_{1,n}), a_{2j_1+2,n}]$ one e-point more than $\mathcal{T}_n(x, 0) = \mathcal{M}_n(x)$ on $[a_{2j_1+1,n}, a_{2j_1+2,n}]$ and on the remaining intervals of $E^1_{l,n}$ and $E_{l,n}$ (arranged in increasing order) $\mathscr{T}_n(x, \alpha_{1,n})$ and $\mathscr{T}_n(x, 0) = \mathscr{M}_n(x)$ have the same number of e-points, respectively.

Case 2: $c_{2j_1,n} + \alpha_{1,n} < a_{2j_1+1,n}(\alpha_{1,n})$ *and* $c_{2j_{\nu^*}-1,n}(\alpha_{1,n}) = a_{2j_{\nu^*},n}$ *or* $c_{2j_{\nu^*},n} = a_{2j_{\nu^*}+1,n}(\alpha_{1,n})$. Again for simplicity of notation let us first assume that $c_{2j_{\nu^*}-1,n}(\alpha_{1,n}) = a_{2j_{\nu^*},n}$ and that no other gap between an a- and a c-interval becomes closed. Then $E(\alpha_{1,n})$ is of the form

$$\begin{aligned} E(\alpha_{1,n}) &= \bigcup_{j=1}^{j_1-1} [a_{2j-1,n}, a_{2j,n}(\alpha_{1,n})] \cup [a_{2j_1-1,n}, a_{2j_1,n}] \\ &\quad \cup [c_{2j_1-1,n}(\alpha_{1,n}), c_{2j_1,n} + \alpha_{1,n}] \cup \bigcup_{\substack{j=j_1+1 \\ j \neq j_{\nu^*}}}^{l} [a_{2j-1,n}(\alpha_{1,n}), a_{2j,n}] \\ &\quad \cup [a_{2j_{\nu^*}-1,n}(\alpha_{1,n}), c_{2j_{\nu^*},n}] \cup \bigcup_{\substack{\nu=2 \\ \nu \neq \nu^*}}^{l'} [c_{2j_\nu-1,n}(\alpha_{1,n}), c_{2j_\nu,n}] \\ &= E^1_{l,n} \cup [c_{2j_1-1,n}(\alpha_{1,n}), c_{2j_1,n} + \alpha_{1,n}] \cup \tilde{C}_{l'-2,n} \\ &=: E^1_{l,n} \cup C^1_{l'-1,n}, \end{aligned}$$

where $\tilde{C}_{l'-2,n} = \bigcup_{\nu=2,\, \nu \neq \nu^*}^{l'} [c_{2j_\nu-1,n}(\alpha_{1,n}), c_{2j_\nu,n}]$. Note that $\mathscr{T}_n(x, \alpha_{1,n})$ is a T-polynomial on $E(\alpha_{1,n})$ and a minimal polynomial on $E^1_{l,n}$ which has on $[a_{2j_{\nu^*}-1,n}(\alpha_{1,n}), c_{2j_{\nu^*},n}]$ one e-point more than $\mathscr{T}_n(x, 0) = \mathscr{M}_n(x)$ on $[a_{2j_{\nu^*}-1,n}, a_{2j_{\nu^*},n}]$ and on the remaining intervals of $E^1_{l,n}$ and $E_{l,n}$ $\mathscr{T}_n(x, \alpha_{1,n})$ and $\mathscr{T}_n(x, 0)$ have the same number of e-points, respectively. Obviously corresponding statements hold in the remaining cases.

After having carried out this procedure for each $n \geqslant n_0$ there is at least one strictly monotone sequence (n_k) such that for each n_k the same gap is closed. Now by what has been said above E^1_{l,n_k} and E_{l,n_k} as well as the minimal polynomials on the both sets of l intervals satisfy the assumptions of Proposition 1.4. Hence

$$E^1_{l,n_k} \xrightarrow[k \to \infty]{} E_l.$$

In the next step we take the sets

$$E(\alpha_{1,n_k}) = E^1_{l,n_k} \cup C^1_{l'-1,n_k}$$

as starting point, which have now $l'-1$ c-intervals and proceed as above. That is we close another gap, such that there will be at most $l'-2$ c-intervals. As before we will arrive at a subsequence (n_{k_j}) of (n_k) and sets of intervals $E(\alpha_{1,n_k}, \alpha_{2,n_{k_j}})$, $j \in \mathbb{N}$, which are of the form

$$E(\alpha_{1,n_k}, \alpha_{2,n_{k_j}}) = E^2_{l,n_{k_j}} \cup C^2_{l'-2,n_{k_j}},$$

where $E^2_{l, n_{k_j}}$ and $E^1_{l, n_{k_j}}$ as well as the minimal polynomials on these both sets of l intervals satisfy the assumption of Proposition 1.4 and thus

$$E^2_{l, n_{k_j}} \xrightarrow[j \to \infty]{} E_l.$$

After at most $l' \leqslant l-1$ steps we arrive at a sequence of sets of l disjoint intervals, say $E^{l'}_{l, n_\nu}$, $\nu \in \mathbb{N}$, with $E^{l'}_{l, n_\nu} \xrightarrow[\nu \to \infty]{} E_l$ and such that for each $\nu \in \mathbb{N}$ the minimal polynomial of degree n_ν on $E^{l'}_{l, n_\nu}$ is a T-polynomial on $E^{l'}_{l, n_\nu}$ (i.e., there appears no c-interval) which proves the theorem. ▮

Let us mention that the method used in the proof is reversible, that is, it could also be used to generate from a T-polynomial a minimal polynomial with "c-intervals".

As I have been informed by V. Totik ([26]), to whom I mentioned the problem of denseness, he has also proved Theorem 2.1 but by a completely different method, more precisely with the help of the so-called balayage. Further he mentioned that he obtained under the usage of the denseness property inequalities of Markov type for several intervals.

Note added in proof. As we have learned, Theorem 2.1 has also been proved in another completely different way with the help of the so-called comb–map by A. B. Bogatyrev in Section 2.2 of Effective computation of Chebyshev polynomials for several intervals, *Math. USSR Sb.* **190** (1999), 1571–1605.

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